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The Clebsch–Gordan coefficient for SU(m/n) Gel'fand basis

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Abstract. The outer-product reduction coefficients (ORC) of the graded state permutation group, which differ from the ORC of the ordinary permutation group only in signs, have been identified with the Clebsch–Gordan coefficients (CGC) for the special Gel'fand basis of the graded unitary group SU(m/n). The CGC for a general Gel'fand basis of SU(m/n) can be easily obtained from those for the special one. Tables of the SU(m/n) CGC are presented which are valid for arbitrary m and n.

1. Introduction

The graded unitary group SU(m/n) has recently become a topic of interest in physics in the context of supersymmetries relating particles with different statistics (Ne'eman 1979, Dondi and Jarvis 1979, 1981, Iachello 1980). The first evidence for the existence of the supersymmetry in nature has been reported in the field of nuclear physics (Iachello 1980, Balantekin *et al* 1981). Many properties of nuclei in the Os–Pt region, including excitation energies, electromagnetic transition rates and transfer reaction intensities, can be described fairly well (within about 30%) by a U(6/4) supersymmetry.

The Casimir operators, representations, and branching rules of the graded unitary group have been studied extensively (Jarvis and Green 1979, Dondi and Jarvis 1981, Balantekin and Bars 1981, Balantekin 1982, Han *et al* 1981, Sun and Han 1981, Chen *et al* 1983b). More recent progress on the Clebsch–Gordan coefficients (CGC) and isoscalar factors (ISF) or the coefficients of fractional parentage (CFP) of the graded unitary group have been sketched previously in Chen *et al* (1983a), followed by detailed expositions on several separate subjects, such as the formulae for the Gel'fand matrix elements of the generators E^{i}_{i-1} of U(m/n) (Chen and Chen 1983), the identification of the $\mathrm{U}(mp+nq/mq+np)\supset \mathrm{U}(m/n)\times \mathrm{U}(p/q)$ CFP with $\mathrm{U}(mn)\supset \mathrm{U}(m)\times\mathrm{U}(n)$ CFP as well as with the permutation group $\mathrm{S}(f)\supset \mathrm{S}(f_1)\times \mathrm{S}(f_2)$ ISF, and the identification of the $\mathrm{U}(m+p/n+q)\supset \mathrm{U}(m/n)\times\mathrm{U}(p/q)$ CFP with the $\mathrm{U}(m+n)\supset \mathrm{U}(m)\times\mathrm{U}(n)$ CFP as well as with the permutation group outer-product ISF for $\mathrm{S}(f)\supset \mathrm{S}(f_1)\times\mathrm{S}(f_2)$, along with the tabulation of the one-body CFP for the aforementioned group chains (Chen *et al* 1983c, d). What the present paper is concerned with is the construction of the CGC for the Gel'fand basis of SU(m/n).

As we know, several methods are available for calculating the CGC of SU(n) in the Gel'fand basis. They mainly fall into the following two categories. One is the unitary group approach (for SU(3): de Swart 1963, McNamee and Chilton 1964, Bickerstaff *et al* 1982, Sun 1980; for SU(4): Haacke *et al* 1976; for SU(n): Baird and Biedenharn 1963), and the other is the permutation group approach based on the duality between the unitary group and the permutation group (Chen *et al* 1978a). A distinguishing feature of the latter approach lies in the fact that it is rank independent. Another advantage is that it turns out to be the most direct way for extending the calculation of the CGC of the ordinary unitary group to that of the graded unitary group.

The kernel of the permutation group approach to the SU(n) CGC is the identification of the quasi-standard basis of the permutation group with the Gel'fand basis of the unitary group (Chen *et al* 1977) and the introduction of a versatile coefficient, the so-called outer-product reduction coefficient (ORC) of the permutation group (Chen *et al* 1978a). It was shown that the ORC is the coupling coefficient for the $U(m+n)\supset U(m)\times U(n)$ irreducible basis (IRB) (Chen *et al* 1978b, 1983c), and the CGC for the special Gel'fand basis of SU(n). Furthermore, from it we can obtain the CFP for $U(m+n)\supset U(m)\times U(n)$ and $U(m/n)\supset U(m)\times U(n)$ etc (Chen *et al* 1983c), as well as the CGC for a general Gel'fand basis of SU(n) (Chen *et al* 1978a). The present work will demonstrate that the ORC for the graded state permutation group, which are the same as the ordinary ORC up to sign factors, are the CGC for the special Gel'fand basis of SU(m/n), and that the CGC for a general Gel'fand basis of SU(m/n) can be obtained from those for the special Gel'fand basis. Tables of the SU(m/n) CGC, valid for arbitrary m and n, and containing the SU(n) CGC as its special case, are presented.

2. The ORC for the graded state permutation group

We shall follow the notation of Chen *et al* (1983c) as closely as possible. The readers are referred to this reference for any unexplained notation in this paper.

The ORC are the coefficients of a unitary matrix which reduces the outer-product of the irreps $[\sigma_1]$ and $[\sigma_2]$ of the permutation groups $S(f_1)$ and $S(f_2)$, respectively, into the direct sum of the irreps $[\sigma]$ of $S(f)$ with $f=f_1+f_2$,

$$[\sigma_1]\otimes[\sigma_2]=\sum_{\sigma}\oplus\{\sigma_1\sigma_2\sigma\}[\sigma], \tag{2.1}$$

where the integers $\{\sigma_1\sigma_2\sigma\}$ are decided by the Littlewood rule. In other words, the ORC are the expansion coefficients for expanding the Yamanouchi basis (YB) $|Y_r^{\sigma}(\omega)\rangle$ of $S(f)$ in terms of the YB $|Y_{r_1}^{\sigma_1}(\omega_1)\rangle$ and $|Y_{r_2}^{\sigma_2}(\omega_2)\rangle$ of $S(f_1)$ and $S(f_2)$ acting on the coordinate indices represented by the normal order sequences (ω_1) and (ω_2), respectively

$$|Y_r^{[\sigma]\theta}(\omega)\rangle=\sum_{r_1r_2\omega_1\omega_2}C^{[\sigma]\theta,r}_{\sigma_1r_1\omega_1,\sigma_2r_2\omega_2}|Y_{r_1}^{\sigma_1}(\omega_1)\rangle|Y_{r_2}^{\sigma_2}(\omega_2)\rangle, \tag{2.2}$$

$$(\omega)=(12\ldots f),\qquad \theta=1,2,\ldots\{\sigma_1\sigma_2\sigma\},$$

where $Y_{r_i}^{\sigma_i}(\omega_i)$ denote the standard Young tableaux of $S(f_i)$ acting on (ω_i). More concisely, (2.2) can be rewritten as

$$|[\sigma]\theta m\rangle=\sum_{m_1m_2}\langle[\sigma]\theta m|\sigma_1m_1\sigma_2m_2\rangle|\sigma_1m_1\rangle|\sigma_2m_2\rangle, \tag{2.3}$$

$$m=r\omega,\qquad m_i=r_i\omega_i.$$

Now let us carry over the discussion of the ORC of the ordinary permutation group $S(f)$ into that of the graded state permutation group $\mathring{\mathscr{S}}(f)$.

Suppose there are $f = m+n$ single particle (SP) states

$$A_i = \begin{cases} a_i, & i = 1, 2, \ldots m, \\ \alpha_{i-m}, & i = m+1, \ldots m+n, \end{cases} \tag{2.4}$$

with a_i and α_j representing the bosonic (commuting) and fermionic (anticommuting) SP states respectively. The ordering of the SP states is specified as $A_1 < A_2 < \ldots < A_f$. A graded state permutation $(A_i A_l)^\circ \in \mathring{\mathscr{S}}(f)$ is defined by its action on f-particle product states (Chen *et al* 1983b):

$$(A_i A_l)^\circ |A_p \ldots A_i A_j \ldots A_k A_l \ldots A_q\rangle$$
$$= \begin{bmatrix} & A_j \\ A_i & \vdots \\ & A_l \end{bmatrix} \begin{bmatrix} & A_j \\ A_l & \vdots \\ & A_k \end{bmatrix} |A_p \ldots A_l A_j \ldots A_k A_i \ldots A_q\rangle, \tag{2.5}$$

where the first and second factors are the sign factors (Jarvis and Green 1979),

$$\begin{bmatrix} & A_j \\ A_i & \vdots \\ & A_k \end{bmatrix} = [A_i A_j] \ldots [A_i A_k], \tag{2.6}$$

and

$$[A_i A_j] = \begin{cases} -1 & \text{for } A_i \text{ and } A_j \text{ being both fermionic} \\ +1 & \text{otherwise.} \end{cases} \tag{2.7}$$

The sign factor of (2.6) comes from the fact that for the SP state A_i in (2.5) to reach to its final position it has to cross over the SP states $A_j, \ldots A_k$.

Instead of grouping the f ordinals into two normal order sequences (ω_1) and (ω_2), we now group the f SP states

$$(\mathring{\omega}) = (A_1 A_2 \ldots A_f) \tag{2.8}$$

into the two normal order states

$$\begin{aligned} &(\mathring{\omega}_i) = (A_1^{(i)} A_2^{(i)} \ldots A_{f_i}^{(i)}), \\ &A_1^{(i)} < A_2^{(i)} < \ldots < A_{f_i}^{(i)}, \end{aligned} \qquad i = 1, 2. \tag{2.9}$$

The YB of $\mathring{\mathscr{S}}(f_i)$ acting on the SP states $(\mathring{\omega}_i)$ can be designated by

$$|\sigma_i m_i\rangle^\circ = |Y_{r_i}^{\sigma_i}(\mathring{\omega}_i)\rangle, \tag{2.10}$$

where $Y_{r_i}^{\sigma_i}(\mathring{\omega}_i)$ are the graded Weyl tableaux resulted from filling the Young diagram Y^{σ_i} with the SP states $(\mathring{\omega}_i)$ according to the ordering specified by the Yamanouchi symbol r_i.

We now claim that the counterpart of (2.3) for the graded state permutation group is

$$|[\sigma]\theta m\rangle^\circ = \sum_{m_1 m_2} [\omega_1, \omega_2] \langle [\sigma]\theta m | \sigma_1 m_1 \sigma_2 m_2 \rangle |\sigma_1 m_1\rangle^\circ |\sigma_2 m_2\rangle^\circ, \tag{2.11}$$

where $[\omega_1, \omega_2]$ are sign factors

$$[\omega_1, \omega_2] = \prod_{\substack{i \in \mathring{\omega}_1, j \in \mathring{\omega}_2 \\ i > j}} \begin{bmatrix} & j_1 \\ i & \vdots \\ & j_p \end{bmatrix} = \prod_{\substack{i \in \mathring{\omega}_1, j \in \mathring{\omega}_2 \\ i > j}} \begin{bmatrix} i_1 & \\ \vdots & j \\ i_q & \end{bmatrix}. \tag{2.12}$$

Otherwise stated, the ORC for the graded state permutation group and the ordinary permutation group are the same except for the difference in signs. To show this, we only need to demonstrate that the basis vector $[\omega_1, \omega_2]|\sigma_1 m_1\rangle^\circ|\sigma_2 m_2\rangle^\circ$ transforms under the graded state permutation $\mathring{\mathscr{P}}$ in exactly the same way as the basis vector $|\sigma_1 m_1\rangle|\sigma_2 m_2\rangle$ under the ordinary permutation p.

The action of a permutation p on a normal order sequence $(\omega_{12}) = (\omega_1, \omega_2)$ can be written as

$$p(\omega_{12}) = (\tilde{\omega}), \tag{2.13a}$$

where $(\tilde{\omega})$ is usually not a normal order sequence, but can be brought to a normal order sequence $(\omega'_{12}) = (\omega'_1, \omega'_2)$ through the permutation

$$p_1 p_2 = \begin{pmatrix} \tilde{\omega} \\ \omega'_{12} \end{pmatrix}, \qquad p_i \in \mathrm{S}(f_i) \text{ acting on } (\omega'_i), \tag{2.13b}$$

i.e.

$$p(\omega_{12}) = p_1 p_2 (\omega'_{12}). \tag{2.13c}$$

Let us introduce the order-preserving permutation

$$Q_{\omega_{12}} = \begin{pmatrix} \omega \\ \omega_{12} \end{pmatrix} = \begin{pmatrix} 12\ldots f \\ \omega_1, \omega_2 \end{pmatrix}, \tag{2.14}$$

which brings the natural sequence $(\omega) = (12\ldots f)$ into the normal order sequence $(\omega_{12}) = (\omega_1, \omega_2)$. Using the order-preserving permutation (2.14), from (2.13c) we have

$$pQ_{\omega_{12}} = p_1 p_2 Q_{\omega'_{12}}. \tag{2.15}$$

Therefore the basis vectors $|\sigma_1 m_1\rangle|\sigma_2 m_2\rangle$ transform as

$$p[|Y^{\sigma_1}_{r_1}(\omega_1)\rangle|Y^{\sigma_2}_{r_2}(\omega_2)\rangle] = \sum_{r'_1 r'_2} D^{\sigma_1}_{r'_1 r_1}(p_1) D^{\sigma_2}_{r'_2 r_2}(p_2)|Y^{\sigma_1}_{r'_1}(\omega'_1)\rangle|Y^{\sigma_2}_{r'_2}(\omega'_2)\rangle, \tag{2.16}$$

where D^{σ_i} are the Young–Yamanouchi matrices.

Now turn to the graded state permutation group. Let us call

$$|\mathring{\omega}\rangle = |A_1 A_2 \ldots A_f\rangle, \qquad |\mathring{\omega}_{12}\rangle = |\mathring{\omega}_1, \mathring{\omega}_2\rangle, \tag{2.17}$$

the natural and normal order state respectively. In parallel to (2.14), the following operator

$$\mathring{Q}_{\omega_{12}} = \begin{pmatrix} \mathring{\omega} \\ \mathring{\omega}_{12} \end{pmatrix} \tag{2.18}$$

is called the order-preserving graded state permutation. It is easily seen that

$$\mathring{Q}_{\omega_{12}}|\mathring{\omega}\rangle = [\omega_1, \omega_2]|\mathring{\omega}_{12}\rangle. \tag{2.19}$$

Due to the isomorphism between $\mathrm{S}(f)$ and $\mathring{\mathscr{S}}(f)$, in analogy with (2.15) we have

$$\mathring{\mathscr{P}}\mathring{Q}_{\omega_{12}} = \mathring{\mathscr{P}}_1 \mathring{\mathscr{P}}_2 \mathring{Q}_{\omega'_{12}}, \qquad \mathscr{P}_i \in \mathring{\mathscr{S}}(f_i) \text{ acting on } (\mathring{\omega}'_i). \tag{2.20}$$

Multiplying (2.20) from the right by $|\mathring{\omega}\rangle$,

$$\mathring{\mathscr{P}}\mathring{Q}_{\omega_{12}}|\mathring{\omega}\rangle = \mathring{\mathscr{P}}_1 \mathring{\mathscr{P}}_2 \mathring{Q}_{\omega'_{12}}|\mathring{\omega}\rangle. \tag{2.21a}$$

Then using (2.19), one has

$$\mathring{\mathscr{P}}[\omega_1, \omega_2]|\mathring{\omega}_{12}\rangle = \mathring{\mathscr{P}}_1 \mathring{\mathscr{P}}_2 [\omega'_1, \omega'_2]|\mathring{\omega}'_{12}\rangle. \tag{2.21b}$$

Hence the transformation law for the basis vectors $[\omega_1, \omega_2]|\sigma_1 m_1\rangle^\circ|\sigma_2 m_2\rangle^\circ$ is

$$\mathring{\mathscr{P}}([\omega_1, \omega_2]|Y^{\sigma_1}_{r_1}(\mathring{\omega}_1)\rangle|Y^{\sigma_2}_{r_2}(\mathring{\omega}_2)\rangle) = \sum_{r_1' r_2'} D^{\sigma_1}_{r_1' r_1}(p_1) D^{\sigma_2}_{r_2' r_2}(p_2)([\omega_1', \omega_2']|Y^{\sigma_1}_{r_1}(\mathring{\omega}_1')\rangle|Y^{\sigma_2}_{r_2}(\mathring{\omega}_2')\rangle). \tag{2.22}$$

Comparing (2.22) with (2.16) shows that (2.11) is correct.

An example of (2.13*c*) and (2.21*b*) is

$p_{36}(134, 256) = (\tilde{\omega}) = (164, 253) = p_{46}p_{35}(146, 235).$

$$(A_3A_6)^\circ \begin{bmatrix} A_2 & A_3 \\ & A_4 \end{bmatrix} |A_1A_3A_4, A_2A_5A_6\rangle = (A_4A_6)^\circ(A_3A_5)^\circ \begin{bmatrix} A_4 & A_2 \\ A_6 & A_3 \end{bmatrix} [A_6A_5]|A_1A_4A_6, A_2A_3A_5\rangle. \tag{2.23}$$

It should be noted that although both the graded coordinate permutation group $\mathring{S}(f)$ and the graded state permutation group $\mathring{\mathscr{S}}(f)$ are isomorphic to the ordinary permutation group $S(f)$, a dissimilarity exists between $\mathring{S}(f)$ and $\mathring{\mathscr{S}}(f)$, i.e. the ORC of $\mathring{S}(f)$ is identical with that of $S(f)$ (Chen *et al* 1983c) but not with that of $\mathring{\mathscr{S}}(f)$ on account of the extra sign factor $[\omega_1, \omega_2]$. In order to fully understand this dissimilarity, it is instructive to supplement the proof on the equality of the ORC of $\mathring{S}(f)$ and that of $S(f)$, which is omitted in the paper of Chen *et al* (1983).

We must point out that an isolated YB of $\mathring{S}(f)$ is meaningless, in contrast to the YB of $\mathring{\mathscr{S}}(f)$, which unambiguously represents a special Gel'fand basis of $SU(m/n)$. The YB of $\mathring{S}(f)$ has a definite meaning only when it is used in conjunction with an IRB of $SU(m/n)$. Let the IRB of $\mathring{S}(f_i)$ and $SU(m/n)$ be denoted by

$$|Y^{\sigma_i}_{r_i}(\omega_i)\rangle^\circ \equiv |Y^{\sigma_i}_{r_i}(\omega_i), W^{\sigma_i}_{s_i}\rangle^\circ = \mathring{P}^{[\sigma_i]s_i}_{r_i}(\omega_i)|A_1^{(i)}A_2^{(i)}\ldots A_{f_i}^{(i)}\rangle \tag{2.24}$$

where equation (30) in Chen *et al* (1983b) has been used, $W^{\sigma_i}_{s_i}$ denotes the Weyl tableau, $\mathring{P}^{[\sigma_i]s_i}_{r_i}(\omega_i)$ is the projection operator of $\mathring{S}(f_i)$ acting on the coordinate indices (ω_i). Whether $|Y^{\sigma_i}_{r_i}(\omega_i)\rangle^\circ$ represents a YB of $\mathring{S}(f)$ or $S(f)$ is entirely decided by whether $W^{\sigma_i}_{s_i}$ is a graded or ordinary Weyl tableau. The imparity of $\mathring{S}(f)$ and $\mathring{\mathscr{S}}(f)$ is in fact a reflection of the difference between the coordinate indices and the state indices, the former being ungraded, while the latter are graded according to their being bosonic or fermionic. Therefore, in essence we only have one graded permutation group, i.e. the graded state permutation group, while the so-called graded coordinate permutation group $\mathring{S}(f)$ is nothing other than the representation of the ordinary permutation group $S(f)$ in a graded space. This point is quite clear from the definition of $\mathring{S}(f)$ (see equation (10) in Chen *et al* (1983b)).

Now let us study the action of the permutation $\mathring{p} \in \mathring{S}(f)$ on the product of the basis vectors of (2.24). In view of the isomorphism between $\mathring{S}(f)$ and $S(f)$, (2.13*c*) remains true for $\mathring{S}(f)$. It thus follows that

$$\mathring{p}\mathring{P}^{[\sigma_1]s_1}_{r_1}(\omega_1)\mathring{P}^{[\sigma_2]s_2}_{r_2}(\omega_2) = \mathring{p}_1\mathring{p}_2 P^{[\sigma_1]s_1}_{r_1}(\omega_1')P^{[\sigma_2]s_2}_{r_2}(\omega_2'). \tag{2.25}$$

Using (2.24) and (2.25), we immediately obtain

$$\mathring{p}|Y^{\sigma_1}_{r_1}(\omega_1)\rangle^\circ|Y^{\sigma_2}_{r_2}(\omega_2)\rangle^\circ = \mathring{p}_1\mathring{p}_2|Y^{\sigma_1}_{r_1}(\omega_1')\rangle^\circ|Y^{\sigma_2}_{r_2}(\omega_2')\rangle^\circ = \sum_{r_1' r_2'} D^{\sigma_1}_{r_1' r_1}(p_1)D^{\sigma_2}_{r_2' r_2}(p_2)|Y^{\sigma_1}_{r_1'}(\omega_1')\rangle^\circ|Y^{\sigma_2}_{r_2'}(\omega_2')\rangle^\circ. \tag{2.26}$$

Comparing (2.26) with (2.16) we see that the ORC of $\mathring{S}(f)$ and $S(f)$ are indeed the same.

3. The CGC of SU(m/n)

According to Chen *et al* (1983b), when there are repeated SP states in $(\mathring{\omega})$, the basis vector $|[\sigma]m\rangle^\circ = |Y_r^\sigma(\mathring{\omega})\rangle$ becomes the un-normalised quasi-standard basis of the graded state permutation group, and the latter has been identified with the Gel'fand basis $|[\sigma]\omega\rangle^\circ$ of SU(m/n),

$$|[\sigma]m\rangle^\circ = \mathring{R}^{[\sigma]m}|[\sigma]w\rangle^\circ, \qquad |[\sigma_i]m_i\rangle^\circ = \mathring{R}^{[\sigma_i]m_i}|[\sigma_i]w_i\rangle^\circ, \tag{3.1}$$

where $|[\sigma]w\rangle^\circ$ and $|[\sigma_i]w_i\rangle^\circ$ are normalised Gel'fand bases of SU(m/n), and

$$\mathring{R}^{[\sigma]m} \equiv \mathring{R}^{[\sigma]r}(\mathring{\omega}), \qquad \mathring{R}^{[\sigma_i]m_i} \equiv \mathring{R}^{[\sigma_i]r_i}(\mathring{\omega}_i), \tag{3.2}$$

are the normalisation constants depending on σ, r, ω and σ_i, r_i, ω_i respectively. The indices w and w_i label the graded Weyl tableaux. Notice that the correspondence between m and w, or m_i and w_i is not one-to-one, instead there may be several $m(m_i)$ corresponding to the same $w(w_i)$.

Equation (2.11) remains valid when some of the SP states in $(\mathring{\omega})$ and $(\mathring{\omega}_i)$ are identical. Inserting (3.1) into (2.11), we have

$$|[\sigma]\theta w\rangle^\circ = (\mathring{R}^{[\sigma]m})^{-1} \times \sum_{w_1 w_2}\left(\sum_{m_1 m_2}{}' [\omega_1, \omega_2]\langle[\sigma]\theta m|\sigma_1 m_1 \sigma_2 m_2\rangle \mathring{R}^{[\sigma_1]m_1}\mathring{R}^{[\sigma_2]m_2}\right)|\sigma_1 w_1\rangle^\circ|\sigma_2 w_2\rangle^\circ \tag{3.3}$$

where the prime in the second summation symbol means that the summation is restricted to those m_i which correspond to the same graded Weyl tableau w_i.

From (3.3) we obtain a relation between the CGC of SU(m/n) and the ORC of S(f):

$${}^\circ\langle[\sigma]\theta w|\sigma_1 w_1 \sigma_2 w_2\rangle^\circ = (\mathring{R}^{[\sigma]m})^{-1} \sum_{m_1 m_2}{}' \mathring{R}^{[\sigma_1]m_1}\mathring{R}^{[\sigma_2]m_2}[\omega_1, \omega_2]\langle[\sigma]\theta m|\sigma_1 m_1 \sigma_2 m_2\rangle, \tag{3.4}$$

where ${}^\circ\langle[\sigma]\theta w|\sigma_1 w_1 \sigma_2 w_2\rangle^\circ$ denotes the CGC of SU(m/n).

3.1. Special cases

3.1.1. The special Gel'fand basis. When all the SP states in $(\mathring{\omega})$ are different, $|[\sigma]m\rangle^\circ$ and $|[\sigma_i]m_i\rangle^\circ$ become the special Gel'fand basis of SU(m/n), and all the norms $\mathring{R}^{[\sigma]m}$ and $\mathring{R}^{[\sigma_i]m_i}$ are equal to one (Chen *et al* 1983b). Now the correspondence between m and w, or m_i and w_i is one-to-one, and (3.1) and (3.4) reduce to

$$|[\sigma]m\rangle^\circ = |[\sigma]w\rangle^\circ, \qquad |[\sigma_i]m_i\rangle^\circ = |[\sigma_i]w_i\rangle^\circ, \tag{3.5}$$

$${}^\circ\langle[\sigma]\theta w|\sigma_1 w_1 \sigma_2 w_2\rangle^\circ = [\omega_1, \omega_2]\langle[\sigma]\theta m|\sigma_1 m_1 \sigma_2 m_2\rangle. \tag{3.6}$$

In other words, the CGC for the special Gel'fand basis of SU(m/n) is equal to the ORC of the graded state permutation group $\mathring{S}(m+n)$.

3.1.2. Totally bosonic case. If all the SP states are bosonic, i.e. $n=0$, then $[\omega_1, \omega_2]=1$; the Gel'fand bases $|[\sigma]w\rangle^\circ$ and $|[\sigma_i]w_i\rangle^\circ$ of SU$(m/0)$ are the Gel'fand bases of the ordinary unitary group SU(m), and the norms $\mathring{R}^{[\sigma]m}$ and $\mathring{R}^{[\sigma_i]m_i}$ for $\mathring{S}(f)$ become the norms $R^{[\sigma]m}$ and $R^{[\sigma_i]m_i}$ for S(f), respectively (Chen *et al* 1978a). In this case, (3.4) reduces to the expression of the SU(m) CGC in terms of the ORC of S(f), first derived by Chen *et al* (1978a).

3.1.3. Totally fermionic case. (a) Without repeated SP states.

In this case, from (2.19) we know that the sign factor

$$[\omega_1, \omega_2] = \delta_{\omega_{12}}, \tag{3.7}$$

where $\delta_{\omega_{12}}$ is the parity associated with the normal order sequence (ω_{12}) and $\delta_{\omega_{12}} = \pm 1$ as the number of transpositions required to change the natural sequence $(12\ldots f)$ into the normal order sequence (ω_{12}) is even or odd.

Attach the quantum numbers for the graded coordinate permutation group $\mathring{S}(f)$ to the bases in (2.11) and using (3.7), we have

$$\left|\begin{matrix}[\sigma]\theta \\ \sigma_1 s_1 \sigma_2 s_2, m\end{matrix}\right\rangle^{\circ} = \sum_{m_1 m_2} \langle[\sigma]\theta m|\sigma_1 m_1 \sigma_2 m_2\rangle \delta_{\omega_{12}} \left|\begin{matrix}\sigma_1 \\ s_1, m_1\end{matrix}\right\rangle^{\circ} \left|\begin{matrix}\sigma_2 \\ s_2, m_2\end{matrix}\right\rangle^{\circ}. \tag{3.8}$$

Recall that $m_i = r_i\omega_i$, and r_i and s_i are the Yamanouchi numbers. The left-hand side of (3.8) is now the $\mathring{S}(f) \supset \mathring{S}(f_1) \times \mathring{S}(f_2)$ IRB and the SU(m/n) Gel'fand basis (Chen *et al* 1983c).

According to (31b) in Chen *et al* (1983b) and (6.5) in Chen *et al* (1983c), the IRB of SU($0/n$) and SU(n) are related as

$$\left|\begin{matrix}[\sigma_i] \\ s_i, m_i\end{matrix}\right\rangle^{\circ} = \Lambda^{\sigma_i}_{s_i} \Lambda^{\sigma_i}_{r_i} \left|\begin{matrix}[\tilde{\sigma}_i] \\ \tilde{s}_i, \tilde{m}_i\end{matrix}\right\rangle, \tag{3.9a}$$

$$\left|\begin{matrix}[\sigma]\theta \\ \sigma_1 s_1 \sigma_2 s_2, m\end{matrix}\right\rangle^{\circ} = \varepsilon(\sigma_1\sigma_2\sigma\theta) \Lambda^{\sigma_1}_{s_1} \Lambda^{\sigma_2}_{s_2} \Lambda^{\sigma}_{r} \left|\begin{matrix}[\tilde{\sigma}]\theta \\ \tilde{\sigma}_1 \tilde{s}_1 \tilde{\sigma}_2 \tilde{s}_2, \tilde{m}\end{matrix}\right\rangle, \tag{3.9b}$$

where $\tilde{m}_i = \tilde{r}_i\omega_i$ and $\tilde{m} = \tilde{r}\omega$, and $\varepsilon(\sigma_1\sigma_2\sigma\theta)$ is a phase factor. On the other hand, according to (Chen *et al* 1978a, Chen and Gao 1981), the ORC has the property that

$$\langle[\sigma]\theta m|\sigma_1 m_1 \sigma_2 m_2\rangle \delta_{\omega_{12}} = \varepsilon(\sigma_1\sigma_2\sigma\theta) \Lambda^{\sigma_1}_{r_1} \Lambda^{\sigma_2}_{r_2} \Lambda^{\sigma}_{r} \langle[\tilde{\sigma}]\theta\tilde{m}|\tilde{\sigma}_1 \tilde{m}_1 \tilde{\sigma}_2 \tilde{m}_2\rangle. \tag{3.9c}$$

Combining (3.8) with (3.9) we get

$$\left|\begin{matrix}[\tilde{\sigma}]\theta \\ \tilde{\sigma}_1 \tilde{s}_1 \tilde{\sigma}_2 \tilde{s}_2, \tilde{m}\end{matrix}\right\rangle = \sum_{m_1 m_2} \langle[\tilde{\sigma}]\theta\tilde{m}|\tilde{\sigma}_1 \tilde{m}_1 \tilde{\sigma}_2 \tilde{m}_2\rangle \left|\begin{matrix}[\tilde{\sigma}_1] \\ \tilde{s}_1, \tilde{m}_1\end{matrix}\right\rangle \left|\begin{matrix}[\tilde{\sigma}_2] \\ \tilde{s}_2, \tilde{m}_2\end{matrix}\right\rangle. \tag{3.10a}$$

Suppressing the quantum numbers for the coordinate permutation group and replacing the summation indices m_1 and m_2 with $\tilde{m}_1$ and $\tilde{m}_2$, it becomes

$$|[\tilde{\sigma}]\theta\tilde{m}\rangle = \sum_{\tilde{m}_1 \tilde{m}_2} \langle[\tilde{\sigma}]\theta\tilde{m}_1|\tilde{\sigma}_1 \tilde{m}_1 \tilde{\sigma}_2 \tilde{m}_2\rangle |\tilde{\sigma}_1 \tilde{m}_1\rangle |\tilde{\sigma}_2 \tilde{m}_2\rangle. \tag{3.10b}$$

Hence we see that this case reduces to the totally bosonic case.

(b) With repeated SP states.

From (44a) in Chen *et al* (1983b), as well as (3.7), (3.9c) and (3.4), we can show that

$${}^{\circ}\langle[\sigma]\theta w|\sigma_1 w_1 \sigma_2 w_2\rangle^{\circ} = \varepsilon\langle[\tilde{\sigma}]\theta\tilde{w}|\tilde{\sigma}_1 \tilde{w}_1 \tilde{\sigma}_2 \tilde{w}_2\rangle \tag{3.11}$$

where ε is a sign factor depending on σ_1, σ_2 and σ as well as on the component indices w_1, w_2 and w. Equation (3.11) shows that the totally fermionic case with repeated SP states again corresponds to the totally bosonic case. For instance, tables 3.3(a), (b), (c) correspond to tables 2.5(a), (b), (c), respectively.

3.2. Phase convention

We use the Baird–Biedenharn (1965) phase convention, i.e. demanding that the CGC of the highest weight (HW) state be real positive,

$$\langle [\sigma]\theta HW | \sigma_1 HW_1, \sigma_2 w_2\rangle > 0. \tag{3.12}$$

This in turn fixes the overall phase of the ORC (Chen and Gao 1981). The Baird–Biedenharn phase convention is a generalisation of the Condon–Shortley phase convention for the SU(2) CGC. For example, tables 2.1, 2.2(a), (b), 2.3(a), (b), (c) are precisely the usual SU(2) CGC.

4. Tables of the SU(m/n) CGC

In this section we present some tables for the SU(m/n) CGC. The tables are classified into three types. All the table headings refer to the graded Weyl tableaux, but for simplicity, we have deleted all the small circles '∘'—the tags for the IRB of the graded unitary group SU(m/n).

4.1. Special Gel'fand basis

From (3.6) and the ORC in Chen and Gao (1981), we immediately obtain the CGC for the SU(m/n) special Gel'fand basis, listed in tables 1.1–1.12 for systems with up to five particles. The table headings have the following meaning

Tables 1.1–1.6

		$(\sigma_1 w_1, \sigma_2 w_2)$
σw	Norm	

Tables 1.7–1.12

		$(\sigma_1 w_1,)$
σw	Norm	

where

$$(\sigma_1 w_1,) \equiv (\sigma_1 w_1, \sigma_2 w_2) \equiv [\omega_1, \omega_2] |\sigma_1 w_1\rangle |\sigma_2 w_2\rangle. \tag{4.1}$$

In tables 1.7–1.12, the special Gel'fand basis of SU(m/n), or the YB of $\mathring{\mathscr{S}}(m+n)$, is labelled, for convenience, by the partition and an ordinal numbering the YB vector in the decreasing page order of the Yamanouchi numbers (Hamermesh 1962). The second column is the normalisation constant. The value listed is the square of the CGC. An asterisk denotes a negative CGC value.

By identifying (123 . . .) with a specific normal order state ($\mathring{\omega}$), we obtain a SU(m/n) table for a specific case. For instance, by letting $1234 = abcd$, $abc\alpha$, $ab\alpha\beta$, $a\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$ we can obtain the SU(m/n) CGC for the cases of 4 bosons, 3 bosons and 1 fermion, . . . down to 4 fermions.

The SU(m/n) CGC for a six-particle system can be found from the ORC of S(6) (Chen and Gao 1981), which, however, are relatively inaccessible. Fortunately, with the S(5) ORC listed in tables 1.7–1.12 and the S(6) ⊃ S(5) outer-product ISF (i.e. the U($m+n$) ⊃ U(m) × U(n) one-body CFP)

$$\left\langle \begin{matrix} \sigma \\ \sigma' \end{matrix} \middle| \begin{matrix} \sigma_1 & \sigma_2 \\ \sigma_1' & \sigma_2' \end{matrix} \right\rangle^{\theta}_{\theta'} \equiv \left\langle \begin{matrix} [\sigma] \\ \sigma_1\sigma_2 \end{matrix} \middle| \begin{matrix} [\sigma'] & [1] \\ \sigma_1'\sigma_2' & \end{matrix} \right\rangle^{\theta}_{\theta'}, \tag{4.2}$$

listed in Chen *et al* (1983c), we can easily reconstruct the S(6) ORC by using the inverse of equation (2.13*a*) in Chen *et al* (1983*c*),

$$\langle[\sigma]\theta m_1|\sigma_1 m_1 \sigma_2 m_2\rangle = \sum_{\theta'} \left\langle \begin{matrix} \sigma \\ \sigma' \end{matrix} \middle| \begin{matrix} \sigma_1 & \sigma_2 \\ \sigma_1' & \sigma_2' \end{matrix} \right\rangle^{\theta}_{\theta'} \langle[\sigma']\theta' m'|\sigma_1' m_1' \sigma_2' m_2'\rangle. \quad (4.3)$$

For example

$$\left\langle \begin{matrix} 134 \\ 26 \\ 5 \end{matrix} \middle| \begin{matrix} 13 & 24 \\ 6 & 5 \end{matrix} \right\rangle^{\theta} = \left\langle \begin{matrix} [321] \\ [311] \end{matrix} \middle| \begin{matrix} [21] & [21] \\ [2] & [21] \end{matrix} \right\rangle^{\theta} \left\langle \begin{matrix} 134 \\ 2 \\ 5 \end{matrix} \middle| \begin{matrix} 13 & \begin{matrix} 24 \\ 5 \end{matrix} \end{matrix} \right\rangle. \quad (4.4)$$

The above example is just the reverse of the example (2.14) in Chen *et al* (1983*c*).

4.2. *Non-special Gel'fand basis for pure bosons or boson–fermion mixture*

From the special CGC listed in table 1 and the norm $\mathring{R}^{[\sigma]m}$ listed in table 2 in Chen *et al* (1983b), or the analytic expression (5.15) in Chen and Chen (1983), we can easily calculate the SU(m/n) CGC for non-special Gel'fand bases. As examples, the non-special SU(m/n) CGC for systems with up to four particles are given in table 2. At the bottom of each table, a prescription is given for identifying the ordinals with the SP states. The first line of each table gives

$$(\sigma_1 w_1, \sigma_2 w_2) = [\omega_1, \omega_2]|\sigma_1 w_1\rangle|\sigma_2 w_2\rangle.$$

For instance, by writing out explicitly, the first line in table 2.2(c) has the following multiple meanings:

(112, 3)	(113, 2)	(123, 1)
$\|112\rangle\|3\rangle$,	$[2, 3]\|113\rangle\|2\rangle$,	$\left[1, \begin{matrix}2\\3\end{matrix}\right]\|123\rangle\|1\rangle$
$\|aab\rangle\|c\rangle$	$\|aac\rangle\|b\rangle$	$\|abc\rangle\|a\rangle$
$\|aab\rangle\|\alpha\rangle$	$\|aa\alpha\rangle\|b\rangle$	$\|ab\alpha\rangle\|a\rangle$
$\|aa\alpha\rangle\|\beta\rangle$	$-\|aa\beta\rangle\|\alpha\rangle$	$\|a\alpha\beta\rangle\|a\rangle$

4.3. *Non-special Gel'fand bases for pure fermions*

The SU($0/n$) CGC can similarly be calculated from (3.4), and are listed in table 3. It is seen that tables 3.1, 3.2 and 3.3 correspond to tables 2.6, 2.4 and 2.5, respectively, in conformity with equation (3.11).

5. Summary and discussion

The CGC for the SU(m/n) Gel'fand basis are simply related to the ORC of the permutation group through equation (3.4). The SU(m/n) CGC for systems with up to six particles can be either found directly from the tables presented in this paper, or calculated from the ORC tables (Chen and Gao 1981) or the outer-product ISF (Chen *et al* 1983c). With the program for the ORC (Chen and Gao 1981) and the analytic expression for the norm $R^{[\sigma]m}$ (Chen and Chen 1983), the SU(m/n) CGC for more complicated cases can be calculated.

Table 1. The SU(m/n) CGC for the special Gel'fand basis.

1.1. [1]⊗[1]=[2]⊕[11]

		(1, 2)	(2, 1)
12	2	1	1
1 2	2	1	*1

1.2. [2]⊗[1]=[3]⊕[21]

		(12, 3)	(13, 2)	(23, 1)
123	3	1	1	1
12 3	6	4	*1	*1
13 2	2		1	*1

1.3. [3]⊗[1]=[4]⊕[31]

		(123, 4)	(124, 3)	(134, 2)	(234, 1)
1234	4	1	1	1	1
123 4	12	9	*1	*1	*1
124 3	6		4	*1	*1
134 2	2			1	*1

1.4. [2]⊗[2]=[4]⊕[31]⊕[22]

		(12, 34)	(13, 24)	(14, 23)	(23, 14)	(24, 13)	(34, 12)
1234	6	1	1	1	1	1	1
123 4	6	1	1	*1	1	*1	*1
124 3	12	4	*1	1	*1	1	*4
134 2	4		1	1	*1	*1	
12 34	12	4	*1	*1	*1	*1	4
13 24	4		1	*1	*1	1	

1.5. $[2]\otimes[11]=[31]\oplus[211]$

		$\left(12, \begin{smallmatrix}3\\4\end{smallmatrix}\right)$	$\left(13, \begin{smallmatrix}2\\4\end{smallmatrix}\right)$	$\left(14, \begin{smallmatrix}2\\3\end{smallmatrix}\right)$	$\left(23, \begin{smallmatrix}1\\4\end{smallmatrix}\right)$	$\left(24, \begin{smallmatrix}1\\3\end{smallmatrix}\right)$	$\left(34, \begin{smallmatrix}1\\2\end{smallmatrix}\right)$
123 4	3	1	1		1		
124 3	24	*4	1	9	1	9	
134 2	8		*1	*1	1	1	4
12 3 4	8	4	*1	1	*1	1	
13 2 4	24		9	*1	*9	1	4
14 2 3	3			1		*1	1

1.6. $[21]\otimes[1]=[31]\oplus[22]\oplus[211]$

		$\left(\begin{smallmatrix}12\\3\end{smallmatrix}, 4\right)$	$\left(\begin{smallmatrix}12\\4\end{smallmatrix}, 3\right)$	$\left(\begin{smallmatrix}13\\4\end{smallmatrix}, 2\right)$	$\left(\begin{smallmatrix}23\\4\end{smallmatrix}, 1\right)$	$\left(\begin{smallmatrix}13\\2\end{smallmatrix}, 4\right)$	$\left(\begin{smallmatrix}14\\2\end{smallmatrix}, 3\right)$	$\left(\begin{smallmatrix}14\\3\end{smallmatrix}, 2\right)$	$\left(\begin{smallmatrix}24\\3\end{smallmatrix}, 1\right)$
123 4	3		1	1	1				
124 3	96	36	4	*1	*1			27	27
134 2	32			1	*1	12	12	3	*3
12 34	16	4	4	*1	*1			*3	*3
13 24	16			3	*3	4	*4	*1	1
12 3 4	32	12	*12	3	3			*1	*1
13 2 4	96			*27	27	36	*4	*1	1
14 2 3	3						1	*1	1

1.7. [4]⊗[1]=[5]⊕[41]

			(1234,)	(1235,)	(1245,)	(1345,)	(2345,)
[5]		5	1	1	1	1	1
[41]	1	20	16	*1	*1	*1	*1
	2	12		9	*1	*1	*1
	3	6			4	*1	*1
	4	2				1	*1

Note: $(1234,) \equiv |1234\rangle|5\rangle$, $(1235,) \equiv [4,5]||1235\rangle|4\rangle$, $(1245,) \equiv \left[3, \begin{smallmatrix}4\\5\end{smallmatrix}\right]|1245\rangle|3\rangle, \ldots$.

1.8. [31]⊗[1]=[41]⊕[32]⊕[311]

			$\binom{123}{4\ ,}$	$\binom{123}{5\ ,}$	$\binom{124}{5\ ,}$	$\binom{134}{5\ ,}$	$\binom{234}{5\ ,}$	$\binom{124}{3\ ,}$	$\binom{125}{3\ ,}$	$\binom{125}{4\ ,}$	$\binom{135}{4\ ,}$	$\binom{235}{4\ ,}$	$\binom{134}{2\ ,}$	$\binom{135}{2\ ,}$	$\binom{145}{2\ ,}$	$\binom{145}{3\ ,}$	$\binom{245}{3\ ,}$
[41]	1	4		1	1	1	1										
	2	540	144	9	*1	*1	*1			128	128	128					
	3	270			4	*1	*1	72	72	8	*2	*2				54	54
	4	90				1	*1				2	*2	24	24	24	6	*6
[32]	1	27	9	9	*1	*1	*1			*2	*2	*2					
	2	432			128	*32	*32	144	*36	*4	1	1				*27	*27
	3	144				32	*32				*1	1	48	*12	*12	*3	3
	4	16							4	4	*1	*1				*3	*3
	5	16									3	*3		4	*4	*1	1
[311]	1	45	18	*18	2	2	2			*1	*1	*1					
	2	1440			*512	128	128	576	*36	*4	1	1				*27	*27
	3	480				*128	128				*1	1	192	*12	*12	*3	3
	4	32							12	*12	3	3				*1	*1
	5	96									*27	27		36	*4	*1	1
	6	3													1	*1	1

$\binom{123}{4\ ,} \equiv \left|\begin{matrix}123\\4\end{matrix}\right\rangle|5\rangle$, $\binom{123}{5\ ,} \equiv [4,5]\left|\begin{matrix}123\\5\end{matrix}\right\rangle|4\rangle, \ldots$

1.9. [22]⊗[1]=[32]⊕[221]

			$\begin{pmatrix}12\\34,\end{pmatrix}$	$\begin{pmatrix}12\\35,\end{pmatrix}$	$\begin{pmatrix}12\\45,\end{pmatrix}$	$\begin{pmatrix}13\\45,\end{pmatrix}$	$\begin{pmatrix}23\\45,\end{pmatrix}$	$\begin{pmatrix}13\\24,\end{pmatrix}$	$\begin{pmatrix}13\\25,\end{pmatrix}$	$\begin{pmatrix}14\\25,\end{pmatrix}$	$\begin{pmatrix}14\\35,\end{pmatrix}$	$\begin{pmatrix}24\\35,\end{pmatrix}$
[32]	1	3			1	1	1					
	2	96		36	4	*1	*1				27	27
	3	32				1	*1		12	12	3	*3
	4	32	16	4	4	*1	*1				*3	*3
	5	32				3	*3	16	4	*4	*1	1
[221]	1	32	16	*4	*4	1	1				3	3
	2	32				*3	3	16	*4	4	1	*1
	3	32		12	*12	3	3				*1	*1
	4	96				*27	27		36	*4	*1	1
	5	3								1	*1	1

$\begin{pmatrix}12\\34,\end{pmatrix} \equiv \left|\begin{matrix}12\\34\end{matrix}\right\rangle |5\rangle$, $\begin{pmatrix}12\\35,\end{pmatrix} = [4, 5] \left|\begin{matrix}12\\35\end{matrix}\right\rangle |4\rangle, \ldots$

1.10. [3]⊗[2]=[5]⊕[41]⊕[32]

			(123,)	(124,)	(134,)	(234,)	(125,)	(135,)	(235,)	(145,)	(245,)	(345,)
[5]		10	1	1	1	1	1	1	1	1	1	1
[41]	1	60	9	9	9	9	*4	*4	*4	*4	*4	*4
	2	36	9	*1	*1	*1	4	4	4	*4	*4	*4
	3	18		4	*1	*1	4	*1	*1	1	1	*4
	4	6			1	*1		1	*1	1	*1	
[32]	1	18	9	*1	*1	*1	*1	*1	*1	1	1	1
	2	36		16	*4	*4	*4	1	1	*1	*1	4
	3	12			4	*4		*1	1	*1	1	
	4	12					4	*1	*1	*1	*1	4
	5	4						1	*1	*1	1	

$(123,) \equiv |123\rangle|45\rangle$, $(124,) \equiv [3, 4]|124\rangle|35\rangle, \ldots$

1.11. [3]⊗[11]=[41]⊕[311]

			(123,)	(124,)	(134,)	(234,)	(125,)	(135,)	(235,)	(145,)	(245,)	(345,)
[41]	1	4	1	1	1	1						
	2	60	*9	1	1	1	16	16	16			
	3	30		*4	1	1	*4	1	1	9	9	
	4	10			*1	1		*1	1	*1	1	4
[311]	1	15	9	*1	*1	*1	1	1	1			
	2	120		64	*16	*16	*4	1	1	9	9	
	3	40			16	*16		*1	1	*1	1	4
	4	8					4	*1	*1	1	1	
	5	24						9	*9	*1	1	4
	6	3								1	*1	1

$(123,) \equiv |123\rangle \left|\begin{matrix}4\\5\end{matrix}\right\rangle$, $(124,) \equiv [3, 4]|124\rangle \left|\begin{matrix}3\\5\end{matrix}\right\rangle, \ldots$

1.12. $[21]\otimes[2]=[41]\oplus[32]\oplus[311]\oplus[221]$

			$\binom{12}{3,}$	$\binom{12}{4,}$	$\binom{13}{4,}$	$\binom{23}{4,}$	$\binom{12}{5,}$	$\binom{13}{5,}$	$\binom{23}{5,}$	$\binom{14}{5,}$	$\binom{24}{5,}$	$\binom{34}{5,}$
[41]	1	6					1	1	1	1	1	1
	2	90		16	16	16	1	1	1	*1	*1	*1
	3	180	36	4	*1	*1	4	*1	*1	1	1	*4
	4	60			1	*1		1	*1	1	*1	
[32]	1	18		1	1	1	1	1	1	*1	*1	*1
	2	576	36	4	*1	*1	64	*16	*16	16	16	*64
	3	192			1	*1		16	*16	16	*16	
	4	192	36	36	*9	*9	16	*4	*4	*4	*4	16
	5	64			9	*9		4	*4	*4	4	
[311]	1	30		3	3	3	*3	*3	*3	3	3	3
	2	960	108	12	*3	*3	*192	48	48	*48	*48	192
	3	320			3	*3		*48	48	*48	48	
	4	64	12	*12	3	3						
	5	192			*27	27						
	6	6										
[221]	1	64	4	4	*1	*1	*16	4	4	4	4	*16
	2	64			3	*3		*12	12	12	*12	
	3	64	12	*12	3	3						
	4	192			*27	27						
	5	6										

			$\binom{13}{2,}$	$\binom{14}{2,}$	$\binom{14}{3,}$	$\binom{24}{3,}$	$\binom{15}{2,}$	$\binom{15}{3,}$	$\binom{25}{3,}$	$\binom{15}{4,}$	$\binom{25}{4,}$	$\binom{35}{4,}$
[41]	1	6										
	2	90								12	12	12
	3	180			27	27		27	27	3	3	*12
	4	60	12	12	3	*3	12	3	*3	3	*3	
[32]	1	18								*3	*3	*3
	2	576			27	27		*108	*108	*12	*12	48
	3	192	12	12	3	*3	*48	*12	12	*12	12	
	4	192			*27	*27						
	5	64	12	*12	*3	3						
[311]	1	30								*1	*1	*1
	2	960			81	81		*36	*36	*4	*4	16
	3	320	36	36	9	*9	*16	*4	4	*4	4	
	4	64			*1	*1		4	4	*4	*4	16
	5	192	36	*4	*1	1	16	4	*4	*36	36	
	6	6		1	*1	1	1	*1	1			
[221]	1	64			*3	*3						
	2	64	4	*4	*1	1						
	3	64			*1	*1		*4	*4	4	4	*16
	4	192	36	*4	*1	1	*16	*4	4	36	*36	
	5	6		1	*1	1	*1	1	*1			

$\binom{12}{3,} \equiv \left|\begin{matrix}12\\3\end{matrix}\right\rangle|45\rangle,\ \binom{12}{4,} \equiv [3,4]\left|\begin{matrix}12\\4\end{matrix}\right\rangle|35\rangle, \ldots$

Table 2. The SU(m/n) CGC for non-special Gel'fand basis.

2.1(a) [2]⊗[1]

	(11, 2)	(12, 1)
112	$\frac{1}{3}$	$\frac{2}{3}$
11 2	$\frac{2}{3}$	$*\frac{1}{3}$

112 = aab, $aa\alpha$

2.1(b)

	(12, 2)	(22, 1)
122	$\frac{2}{3}$	$\frac{1}{3}$
12 2	$\frac{1}{3}$	$*\frac{2}{3}$

122 = abb

2.2(a) [3]⊗[1]

	(111, 2)	(112, 1)
1112	$\frac{1}{4}$	$\frac{3}{4}$
111 2	$\frac{3}{4}$	$*\frac{1}{4}$

1112 = $aaab$, $aaa\alpha$

2.2(b)

	(112, 2)	(122, 1)
1122	$\frac{1}{2}$	$\frac{1}{2}$
112 2	$\frac{1}{2}$	$*\frac{1}{2}$

1122 = $aabb$

2.2(c)

	(112, 3)	(113, 2)	(123, 1)
1123	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
112 3	$\frac{3}{4}$	$*\frac{1}{12}$	$*\frac{1}{6}$
113 2	0	$\frac{2}{3}$	$*\frac{1}{3}$

1123 = $aabc$, $aab\alpha$, $aa\alpha\beta$

2.2(d)

	(122, 3)	(123, 2)	(223, 1)
1223	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
122 3	$\frac{3}{4}$	$*\frac{1}{6}$	$*\frac{1}{12}$
123 2	0	$\frac{1}{3}$	$*\frac{2}{3}$

1223 = $abbc$, $abb\alpha$

2.2(e)

	(123, 3)	(133, 2)	(233, 1)
1233	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
123 3	$\frac{1}{2}$	$*\frac{1}{4}$	$*\frac{1}{4}$
133 2	0	$\frac{1}{2}$	$*\frac{1}{2}$

1233 = $abcc$

2.3(a) [2]⊗[2]

	(11, 12)	(12, 11)
1112	$\frac{1}{2}$	$\frac{1}{2}$
111 2	$\frac{1}{2}$	$*\frac{1}{2}$

1112 = $aaab$, $aaa\alpha$

2.3(b)

	(12, 22)	(22, 12)
1222	$\frac{1}{2}$	$\frac{1}{2}$
122 2	$\frac{1}{2}$	$*\frac{1}{2}$

1222 = $abbb$

2.3(c)

	(11, 22)	(12, 12)	(22, 11)
1122	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
112 2	$\frac{1}{2}$	0	$*\frac{1}{2}$
11 22	$\frac{1}{3}$	$*\frac{1}{3}$	$\frac{1}{3}$

1122 = $aabb$

2.3(d)

	(11, 23)	(12, 13)	(13, 12)	(23, 11)
1123	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
112 3	$\frac{1}{6}$	$\frac{1}{3}$	$*\frac{1}{3}$	$*\frac{1}{6}$
113 2	$\frac{1}{3}$	$*\frac{1}{6}$	$\frac{1}{6}$	$*\frac{1}{3}$
11 23	$\frac{1}{3}$	$*\frac{1}{6}$	$*\frac{1}{6}$	$\frac{1}{3}$

1123 = $aabc$, $aab\alpha$, $aa\alpha\beta$

2.3(*e*)

	(12, 23)	(13, 22)	(22, 13)	(23, 12)
1223	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
122 3	$\frac{1}{3}$	$*\frac{1}{6}$	$\frac{1}{6}$	$*\frac{1}{3}$
123 2	$\frac{1}{6}$	$\frac{1}{3}$	$*\frac{1}{3}$	$*\frac{1}{6}$
12 23	$\frac{1}{6}$	$*\frac{1}{3}$	$*\frac{1}{3}$	$\frac{1}{6}$

1223 = *abbc*, *abbα*

2.3(*f*)

	(12, 33)	(13, 23)	(23, 13)	(33, 12)
1233	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
123 3	$\frac{1}{2}$	0	0	$*\frac{1}{2}$
133 2	0	$\frac{1}{2}$	$*\frac{1}{2}$	0
12 33	$\frac{1}{3}$	$*\frac{1}{6}$	$*\frac{1}{6}$	$\frac{1}{3}$

1233 = *abcc*

2.4(*a*) [2]⊗[11]

	$\left(11, {2 \atop 3}\right)$	$\left(12, {1 \atop 3}\right)$	$\left(13, {1 \atop 2}\right)$
112 3	$\frac{1}{3}$	$\frac{2}{3}$	0
113 2	$*\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$
11 2 3	$\frac{1}{2}$	$*\frac{1}{4}$	$\frac{1}{4}$

1123 = *aabc*, *aabα*, *aaαβ*

2.4(*b*)

	$\left(12, {2 \atop 3}\right)$	$\left(22, {1 \atop 3}\right)$	$\left(23, {1 \atop 2}\right)$
122 3	$\frac{2}{3}$	$\frac{1}{3}$	0
123 2	$*\frac{1}{12}$	$\frac{1}{6}$	$\frac{3}{4}$
12 2 3	$\frac{1}{4}$	$*\frac{1}{2}$	$\frac{1}{4}$

1223 = *abbc*, *abbα*

2.4(*c*)

	$\left(13, {2 \atop 3}\right)$	$\left(23, {1 \atop 3}\right)$	$\left(33, {1 \atop 2}\right)$
123 3	$\frac{1}{2}$	$\frac{1}{2}$	0
133 2	$*\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
13 2 3	$\frac{1}{4}$	$*\frac{1}{4}$	$\frac{1}{2}$

1233 = *abcc*

2.5(*a*) [21]⊗[1]

	$\left({11 \atop 2}, 3\right)$	$\left({11 \atop 3}, 2\right)$	$\left({12 \atop 3}, 1\right)$	$\left({13 \atop 2}, 1\right)$
112 3	0	$\frac{1}{3}$	$\frac{2}{3}$	0
113 2	$\frac{3}{8}$	$\frac{1}{24}$	$*\frac{1}{48}$	$\frac{9}{16}$
11 2 3	$\frac{3}{8}$	$*\frac{3}{8}$	$\frac{3}{16}$	$*\frac{1}{16}$
11 23	$\frac{1}{4}$	$\frac{1}{4}$	$*\frac{1}{8}$	$*\frac{3}{8}$

1123 = *aabc*, *aabα*, *aaαβ*

2.5(*b*)

	$\left(\begin{smallmatrix}12\\2\end{smallmatrix},3\right)$	$\left(\begin{smallmatrix}12\\3\end{smallmatrix},2\right)$	$\left(\begin{smallmatrix}22\\3\end{smallmatrix},1\right)$	$\left(\begin{smallmatrix}13\\2\end{smallmatrix},2\right)$
122 3	0	$\frac{2}{3}$	$\frac{1}{3}$	0
123 2	$\frac{3}{8}$	$\frac{1}{48}$	$*\frac{1}{24}$	$\frac{9}{16}$
12 2 3	$\frac{3}{8}$	$*\frac{3}{16}$	$\frac{3}{8}$	$*\frac{1}{16}$
12 23	$\frac{1}{4}$	$\frac{1}{8}$	$*\frac{1}{4}$	$*\frac{3}{8}$

1223 = *abbc*, *abbα*

2.5(*c*)

	$\left(\begin{smallmatrix}12\\3\end{smallmatrix},3\right)$	$\left(\begin{smallmatrix}13\\2\end{smallmatrix},3\right)$	$\left(\begin{smallmatrix}13\\3\end{smallmatrix},2\right)$	$\left(\begin{smallmatrix}23\\3\end{smallmatrix},1\right)$
123 3	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
133 2	0	$\frac{3}{4}$	$\frac{1}{8}$	$*\frac{1}{8}$
13 2 3	0	$\frac{1}{4}$	$*\frac{3}{8}$	$\frac{3}{8}$
12 33	$\frac{1}{2}$	0	$*\frac{1}{4}$	$*\frac{1}{4}$

1233 = *abcc*

2.6(*a*) [11]⊗[11]

	$\begin{pmatrix}1&1\\2,&3\end{pmatrix}$	$\begin{pmatrix}1&1\\3,&2\end{pmatrix}$
11 23	$\frac{1}{2}$	$\frac{1}{2}$
11 2 3	$\frac{1}{2}$	$*\frac{1}{2}$

1123 = *aabc*, *aabα*, *aaαβ*

2.6(*b*)

	$\begin{pmatrix}1&2\\2,&3\end{pmatrix}$	$\begin{pmatrix}2&1\\3,&2\end{pmatrix}$
12 23	$\frac{1}{2}$	$\frac{1}{2}$
12 2 3	$\frac{1}{2}$	$*\frac{1}{2}$

1223 = *abbc*, *abbα*

2.6(*c*)

	$\begin{pmatrix}1&2\\3,&3\end{pmatrix}$	$\begin{pmatrix}2&1\\3,&3\end{pmatrix}$
12 33	$\frac{1}{2}$	$\frac{1}{2}$
13 2 3	$\frac{1}{2}$	$*\frac{1}{2}$

1233 = *abcc*

Table 3. The SU(0/*n*) CGC for non-special Gel'fand basis.

3.1(*a*) [2]⊗[2]

	$\alpha\beta, \alpha\gamma$	$\alpha\gamma, \alpha\beta$
$\alpha\beta\gamma$ α	$*\frac{1}{2}$	$\frac{1}{2}$
$\alpha\beta$ $\alpha\gamma$	$*\frac{1}{2}$	$*\frac{1}{2}$

3.1(*b*)

	$\alpha\beta, \beta\gamma$	$\beta\gamma, \alpha\beta$
$\alpha\beta\gamma$ β	$\frac{1}{2}$	$*\frac{1}{2}$
$\alpha\beta$ $\beta\gamma$	$\frac{1}{2}$	$\frac{1}{2}$

3.1(*c*)

	$\alpha\gamma, \beta\gamma$	$\beta\gamma, \alpha\gamma$
$\alpha\beta\gamma$ γ	$*\frac{1}{2}$	$\frac{1}{2}$
$\alpha\gamma$ $\beta\gamma$	$*\frac{1}{2}$	$*\frac{1}{2}$

3.2(*a*) [2]⊗[11]

	$\alpha\beta, \begin{smallmatrix}\alpha\\\gamma\end{smallmatrix}$	$\alpha\gamma, \begin{smallmatrix}\alpha\\\beta\end{smallmatrix}$	$\beta\gamma, \begin{smallmatrix}\alpha\\\alpha\end{smallmatrix}$
$\alpha\beta\gamma$ α	$\frac{1}{4}$	$*\frac{1}{4}$	$\frac{1}{2}$
$\alpha\beta$ α γ	$*\frac{3}{4}$	$*\frac{1}{12}$	$\frac{1}{6}$
$\alpha\gamma$ α β	0	$\frac{2}{3}$	$\frac{1}{3}$

3.2(*b*)

	$\alpha\beta, \begin{smallmatrix}\beta\\\gamma\end{smallmatrix}$	$\alpha\gamma, \begin{smallmatrix}\beta\\\beta\end{smallmatrix}$	$\beta\gamma, \begin{smallmatrix}\alpha\\\beta\end{smallmatrix}$
$\alpha\beta\gamma$ β	$*\frac{1}{4}$	$\frac{2}{4}$	$*\frac{1}{4}$
$\alpha\beta$ β γ	$\frac{3}{4}$	$\frac{1}{6}$	$*\frac{1}{12}$
$\alpha\gamma$ β β	0	$\frac{1}{3}$	$\frac{2}{3}$

3.2(c)

	$\alpha\beta,{}^{\gamma}_{\gamma}$	$\alpha\gamma,{}^{\beta}_{\gamma}$	$\beta\gamma,{}^{\alpha}_{\gamma}$
$\begin{smallmatrix}\alpha\beta\gamma\\ \gamma\end{smallmatrix}$	$\frac{1}{2}$	$*\frac{1}{4}$	$\frac{1}{4}$
$\begin{smallmatrix}\alpha\beta\\ \gamma\\ \gamma\end{smallmatrix}$	$\frac{1}{2}$	$\frac{1}{4}$	$*\frac{1}{4}$
$\begin{smallmatrix}\alpha\gamma\\ \beta\\ \gamma\end{smallmatrix}$	0	$*\frac{1}{2}$	$*\frac{1}{2}$

3.3(a) [21]⊗[1]

	$\begin{smallmatrix}\alpha\beta\\ \alpha\end{smallmatrix},\gamma$	$\begin{smallmatrix}\alpha\gamma\\ \alpha\end{smallmatrix},\beta$	$\begin{smallmatrix}\alpha\beta\\ \gamma\end{smallmatrix},\alpha$	$\begin{smallmatrix}\alpha\gamma\\ \beta\end{smallmatrix},\alpha$
$\begin{smallmatrix}\alpha\beta\gamma\\ \alpha\end{smallmatrix}$	$\frac{3}{8}$	$*\frac{3}{8}$	$\frac{1}{16}$	$\frac{3}{16}$
$\begin{smallmatrix}\alpha\beta\\ \alpha\\ \gamma\end{smallmatrix}$	$\frac{3}{8}$	$\frac{1}{24}$	$*\frac{9}{16}$	$*\frac{1}{48}$
$\begin{smallmatrix}\alpha\gamma\\ \alpha\\ \beta\end{smallmatrix}$	0	$*\frac{1}{3}$	0	$*\frac{2}{3}$
$\begin{smallmatrix}\alpha\beta\\ \alpha\gamma\end{smallmatrix}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{8}$	$*\frac{1}{8}$

3.3(b)

	$\begin{smallmatrix}\alpha\beta\\ \beta\end{smallmatrix},\gamma$	$\begin{smallmatrix}\alpha\beta\\ \gamma\end{smallmatrix},\beta$	$\begin{smallmatrix}\alpha\gamma\\ \beta\end{smallmatrix},\beta$	$\begin{smallmatrix}\beta\gamma\\ \beta\end{smallmatrix},\alpha$
$\begin{smallmatrix}\alpha\beta\gamma\\ \beta\end{smallmatrix}$	$\frac{3}{8}$	$*\frac{1}{16}$	$\frac{3}{16}$	$*\frac{3}{8}$
$\begin{smallmatrix}\alpha\beta\\ \beta\\ \gamma\end{smallmatrix}$	$\frac{3}{8}$	$\frac{9}{16}$	$*\frac{1}{48}$	$\frac{1}{24}$
$\begin{smallmatrix}\alpha\gamma\\ \beta\\ \beta\end{smallmatrix}$	0	0	$*\frac{2}{3}$	$*\frac{1}{3}$
$\begin{smallmatrix}\alpha\beta\\ \beta\gamma\end{smallmatrix}$	$\frac{1}{4}$	$*\frac{3}{8}$	$*\frac{1}{8}$	$\frac{1}{4}$

3.3(c)

	$\begin{smallmatrix}\alpha\beta\\ \gamma\end{smallmatrix},\gamma$	$\begin{smallmatrix}\alpha\gamma\\ \beta\end{smallmatrix},\gamma$	$\begin{smallmatrix}\alpha\gamma\\ \gamma\end{smallmatrix},\beta$	$\begin{smallmatrix}\beta\gamma\\ \gamma\end{smallmatrix},\alpha$
$\begin{smallmatrix}\alpha\beta\gamma\\ \gamma\end{smallmatrix}$	$*\frac{1}{4}$	0	$\frac{3}{8}$	$*\frac{3}{8}$
$\begin{smallmatrix}\alpha\beta\\ \gamma\\ \gamma\end{smallmatrix}$	$\frac{3}{4}$	0	$\frac{1}{8}$	$*\frac{1}{8}$
$\begin{smallmatrix}\alpha\gamma\\ \beta\\ \gamma\end{smallmatrix}$	0	$\frac{1}{2}$	$*\frac{1}{4}$	$*\frac{1}{4}$
$\begin{smallmatrix}\alpha\gamma\\ \beta\gamma\end{smallmatrix}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The values of the SU(m/n) CGC are m and n independent. Therefore, each CGC table in fact represents infinitely many tables of the same class. For example, in tables 1.3–1.6 by letting $1234 = abcd$, $abce$, $abde$, $acde$, $bcde, \ldots abc\alpha$, $abc\beta, \ldots abd\alpha, \ldots ab\alpha\beta$, $ab\alpha\gamma, \ldots$, we can get an infinite number of CGC tables.

The Baird–Biedenharn phase convention (3.12) ensures that the CGC which are equivalent under SU(m) will have the same phase. For instance, in table 1.6, we let

$$1234 = aabb, \begin{smallmatrix}123\\ 4\end{smallmatrix} \to \begin{smallmatrix}aab\\ b\end{smallmatrix}, \quad \begin{smallmatrix}12\\ 34\end{smallmatrix} \to \begin{smallmatrix}aa\\ bb\end{smallmatrix}.$$

Using (3.4) and table 1.6, as well as table 2 in Chen *et al* (1983b), we obtain the following table for the U(2) CGC

	$\begin{smallmatrix}aa\\ b\end{smallmatrix},b$	$\begin{smallmatrix}ab\\ b\end{smallmatrix},a$
$\begin{smallmatrix}aab\\ b\end{smallmatrix}$	$\frac{1}{2}$	$\frac{1}{2}$
$\begin{smallmatrix}aa\\ bb\end{smallmatrix}$	$\frac{1}{2}$	$*\frac{1}{2}$

On restricting U(2) to SU(2), the above table becomes identical to table 1.1. Analogously, in table 1.11, we let

$$12345 = aabbc, [41]\,1 = \begin{matrix}1234\\5\end{matrix} \to \begin{matrix}aabb\\c\end{matrix}, [41]\,2 = \begin{matrix}1235\\4\end{matrix} \to \begin{matrix}aabc\\b\end{matrix}, [311]\,1 = \begin{matrix}123\\4\\5\end{matrix} \to \begin{matrix}aab\\b\\c\end{matrix}$$

Using the norm $R^{[\sigma]m}$ for S(2)–S(5) in Chen and Gao (1981), and (3.4) we obtain the following table for the U(3) CGC

	$aab, \begin{matrix}b\\c\end{matrix}$	$abb, \begin{matrix}a\\c\end{matrix}$	$abc, \begin{matrix}a\\b\end{matrix}$
$\begin{matrix}aabb\\c\end{matrix}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\begin{matrix}aabc\\b\end{matrix}$	$*\frac{1}{10}$	$\frac{1}{10}$	$\frac{4}{5}$
$\begin{matrix}aab\\b\\c\end{matrix}$	$\frac{2}{5}$	$*\frac{2}{5}$	$\frac{1}{5}$

On restricting U(3) to SU(3), $\left|\begin{matrix}aab\\b\\c\end{matrix}\right\rangle \to |ab\rangle$.

Note added in proof. Although under the Baird–Biedenharn phase convention, the CGC of U(m) and SU(m) have been identified, the relation between the CGC of U(m/n) and SU(m/n) is not yet clear. Therefore it is safer to replace all the terms 'the CGC of SU(m/n)' in the present paper by 'the CGC of U(m/n)'.

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